1.7 Homogeneous Functions

- ▶ Homogeneous function-I: 2 variable and degree n: $f(tx, ty) = t^n f(x, y)$.
- ▶ Example: $f(x,y) = \frac{x^2+y^2}{x^3+y^3} \Rightarrow f(tx,ty) = \frac{1}{t}\frac{x^2+y^2}{x^3+y^3} = t^{-1}f(x,y) \Rightarrow Degree = -1$.
- ▶ Homogeneous function-I: 3 variable and degree n: $f(tx, ty, tz) = t^n f(x, y, z)$.
- ▶ Example: $f(x,y,z) = \sin(\frac{x+y}{z}) \Rightarrow f(tx,ty,tz) = \sin(\frac{x+y}{z}) = t^0 f(x,y,z) \Rightarrow Degree = 0$.

- ▶ Homogeneous function-II: 2 variable and degree n: $f(x,y) = x^n\phi(\frac{y}{x})$ or, $y^n\psi(\frac{x}{y})$.
- ▶ Example: $f(x,y) = \frac{x-y}{x^3+y^3} = x^{-2}\frac{1-\frac{y}{x}}{1+(\frac{y}{x})^3} = x^{-2}\phi(\frac{y}{x}) \Rightarrow Degree = -2$.
- ▶ Homogeneous function-II: 3 variable and degree n: $f(x,y,z) = x^n\phi(\frac{y}{x}, \frac{z}{x})$.
- ▶ Example: $f(x,y,z) = x^2 + yz + z^2 = x^2[1 + \frac{y}{x}\frac{z}{x} + (\frac{z}{x})^2] = x^2\phi(\frac{y}{x}, \frac{z}{x}) \Rightarrow Degree = 2$.

Theorem 1.8. *Euler's theorem (3 variables): If f is a differentiable homogeneous function of degree n for (x,y,z), then* $\boxed{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf}$.

Proof. Let us consider the function $F(x,y,z,t) = t^{-n}f(tx,ty,tz)$.
Put $u = tx,\ v = ty,\ w = tz$ and differentiate F with respect to t we get,

$$\frac{\partial F}{\partial t} = -nt^{-n-1}f(u,v,w) + t^{-n}(\frac{\partial f}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial t})$$
$$= -nt^{-n-1}f(u,v,w) + t^{-n}(x\frac{\partial f}{\partial u} + y\frac{\partial f}{\partial v} + z\frac{\partial f}{\partial w})$$

Now, if f is homogeneous $\Rightarrow F$ is independent of $t \Rightarrow \frac{\partial F}{\partial t} = 0$.

$$\text{Therefore, } nt^{-n-1}f(u,v,w) = t^{-n}(x\frac{\partial f}{\partial u} + y\frac{\partial f}{\partial v} + z\frac{\partial f}{\partial w})$$
$$\Rightarrow nf(u,v,w) = tx\frac{\partial f}{\partial u} + ty\frac{\partial f}{\partial v} + tz\frac{\partial f}{\partial w}$$
$$\Rightarrow u\frac{\partial f}{\partial u} + v\frac{\partial f}{\partial v} + w\frac{\partial f}{\partial w} = nf(u,v,w)$$

When $t = 1 \Rightarrow u = x,\ v = y,\ w = z \Rightarrow x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf(x,y,z)$.

■ Converse: If $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf(x,y,z)$ holds for all (x,y,z) then f will be a homogeneous function of x,y,z of degree n.
$\Rightarrow$ Let $u = tx,\ v = ty,\ w = tz$. So we have,
$\frac{d}{dt}f(tx,ty,tz) = \frac{d}{dt}f(u,v,w) = \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial t} = x\frac{\partial f}{\partial u} + y\frac{\partial f}{\partial v} + z\frac{\partial f}{\partial w} = \frac{1}{t}nf(u,v,w)$
$\Rightarrow \frac{df}{f} = n\frac{dt}{t} \Rightarrow f = At^n \Rightarrow f(u,v,w) = At^n \Rightarrow f(x,y,z) = A,\ [put,\ t = 1]$.
Therefore, $f(u,v,w) = t^n f(x,y,z) \Rightarrow f(tx,ty,tz) = t^n f(x,y,z)$.
It implies f is a homogeneous function of degree n. □

Example 1.26. *If $u(x,y)$ be a homogeneous function of degree n then show that*

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

$\Rightarrow$ *Since* $u(x,y)$ *is a homogeneous function of degree* $n \Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$ ... (1)
Differentiate partially (1) *with respect to 'x' we get,*
$\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\partial y} = n\frac{\partial u}{\partial x} \Rightarrow x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\partial y} = (n-1)\frac{\partial u}{\partial x}$ ... (2)
Differentiate partially (1) *with respect to 'y' we get,*
$x\frac{\partial^2 u}{\partial y\partial x} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = n\frac{\partial u}{\partial y} \Rightarrow x\frac{\partial^2 u}{\partial x\partial y} + y\frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y}$ ... (3)
Now $(2)\times x \;+\; (3)\times y$ *we get,*
$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (n-1)(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}) \Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$

■ Note: $x^2\frac{\partial^2}{\partial x^2} + 2xy\frac{\partial^2}{\partial x\partial y} + y^2\frac{\partial^2}{\partial y^2} = (x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})^2.$

[Do It Yourself] **1.55.** *If* $u = \tan^{-1}\frac{x^3+y^3}{x-y}$, *then show that* $xu_x + yu_y = \sin 2u$.
$[Hint:\ \tan u \text{ is a homogeneous function of degree } 2]$

[Do It Yourself] **1.56.** *If* $u = \cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, *then show that* $xu_x + yu_y + \frac{1}{2}\cot u = 0$.

1.7.1 Jacobian

If $f_1(x_1,\cdots,x_n)$, $f_2(x_1,\cdots,x_n)$, $\cdots$, $f_n(x_1,\cdots,x_n)$ are functions of $x_1,\cdots,x_n$ then Jacobian of $f_1, f_2,\cdots,f_n$ with respect to $x_1, x_2,\cdots,x_n$ is

$$J = J\left(\frac{f_1,\cdots,f_n}{x_1,\cdots,x_n}\right) = \frac{\partial(f_1,\cdots,f_n)}{\partial(x_1,\cdots,x_n)} = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix}.$$

▶ If $J = \frac{\partial(f_1,\cdots,f_n)}{\partial(x_1,\cdots,x_n)} = 0 \Rightarrow f_1,\cdots,f_n$ are functionally related.

[Do It Yourself] **1.58.** *If* $x = r\cos\theta,\ y = r\sin\theta$, *then show that* $J = \frac{\partial(x,y)}{\partial(r,\theta)} = r$.

[Do It Yourself] **1.59.** *If* $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$, *then show that* $J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$.

[Do It Yourself] **1.60.** *Using Jacobian show that* $u = x+y+z,\ v = xy+yz+zx,\ w = x^3+y^3+z^3-3xyz$ *are functionally dependent. Find the relation.*

1.8 Direction of Curvature

▶ Concave Upwards or, Convex: A curve is said to be concave upwards at a point P when in the immediate neighborhood of P it lies wholly above the tangent at P.

▶ For a Concave upward curve $y = f(x)$ the slope increases i.e. $\boxed{\frac{d^2y}{dx^2} > 0}$.

▶ Concave Downwards or, Concave: A curve is said to be concave downwards at a point P when in the immediate neighborhood of P it lies wholly below the tangent at P.

▶ For a Concave downward curve $y = f(x)$ the slope decreases i.e. $\boxed{\frac{d^2y}{dx^2} < 0}$.

▶ A Point of Inflexion is a point P where $\frac{d^2y}{dx^2}$ changes sign. The curve being concave upwards on one side of this point, and concave downwards on the other.

▶ For the Point of Inflexion P of the curve $y = f(x)$ implies $\boxed{\frac{d^2y}{dx^2} = 0,\ \frac{d^3y}{dx^3} \neq 0}$. If $\frac{dy}{dx} = \infty$ at P, then the conditions are $\boxed{\frac{d^2x}{dy^2} = 0,\ \frac{d^3x}{dy^3} \neq 0}$.

▶ For a Convex or, Concave at P w.r.t. $x - axis$ if $\boxed{y\frac{d^2y}{dx^2} > 0,\ or,\ < 0}$. Ex. $y = x^2$, $y = -x^2$.

▶ For a Convex or, Concave at P w.r.t. $y - axis$ if $\boxed{x\frac{d^2x}{dy^2} > 0,\ or,\ < 0}$. Ex. $x = y^2$, $x = -y^2$.

1.8.1 Problems on Concavity, Convexity and POI

Example 1.29. *Show that the curve $y^3 = 8x^2$ is concave to the foot of the ordinate everywhere (i.e. w.r.t. x axis, try to visualize) except origin.*
$\Rightarrow$ *The given curve is* $y^3 = 8x^2 \Rightarrow y = 2x^{2/3} \Rightarrow y\frac{d^2y}{dx^2} = -\frac{8}{9x^{2/3}}$.
Now $y\frac{d^2y}{dx^2} < 0,\ \forall x \neq 0$. *Therefore the curve is concave to the foot of the ordinate everywhere except origin.*

[Do It Yourself] 1.62. *Show that the curve $y = \ln x$ is convex to the foot of the ordinate in the region $0 < x < 1$ and concave for $x > 1$. Also show that the curve is convex everywhere to the $y - axis$.*

Example 1.30. *Show that the points of inflexion of the curve $y^2 = (x-a)^2(x-b)$ lie on the line $3x + a = 4b$.*

$\Rightarrow$ *The curve is* $y^2 = (x-a)^2(x-b)$ *or,* $y = \pm(x-a)\sqrt{x-b}$.
We can easily check that, $\frac{dy}{dx} = \pm\frac{3x-2b-a}{2\sqrt{x-b}}$, $\frac{d^2y}{dx^2} = \pm\frac{3x-4b+a}{4(x-b)^{3/2}}$ *and* $\frac{d^3y}{dx^3} = \mp\frac{3(x+a-2b)}{8(x-b)^{5/2}}$.
Now $\frac{d^2y}{dx^2} = 0 \Rightarrow 3x - 4b + a = 0 \Rightarrow x = \frac{4b-a}{3}$.
Also at $x = \frac{4b-a}{3}$, $\frac{d^3y}{dx^3} \neq 0$.
The inflexion point are $(\frac{4b-a}{3}, \pm\frac{4}{3\sqrt{3}}(b-a)^{3/2})$ *and POI lies on the line* $3x + a = 4b$.

[Do It Yourself] 1.64. *Show that POI of the curve $y = x\sin x$ lie on the curve $y^2(4 + x^2) = 4x^2$.*

[Do It Yourself] 1.65. *Show that every point in which the curve $y = c\sin\frac{x}{a}$ meets the $x - axis$ is a POI.*

1.8.2 Singular Points

▶ Singular Point: If two or more branches of a curve pass through a point then the point is called a singular point.
▶ Condition: A point (a, b) on a curve $f(x, y) = 0$ is singular if $f_x(a, b) = f_y(a, b) = 0$.
▶ If two (three) branches of a curve pass through a point then the point is called a double (triple) point.
▶ Double Point: $(x^2 + y^2)^2 = 4(x^2 - y^2) \rightleftarrows r^2 = 4\cos 2\theta$. (For the time being you can use the app 'Grapher Free' and visualize the graphs)
▶ Triple Point: $(x^2 + y^2)^2 = 2(x^3 - 3xy^2) \rightleftarrows r = 2\cos 3\theta$.
▶ Quadruple Point: $(x^2 + y^2)^3 = 4(x^2 - y^2)^2 \rightleftarrows r = 2\cos 2\theta$.
▶ If m branches pass through a point then the point is called a multiple point of order m.
▶ Isolated Point or, Acnode: If (x, y) satisfy the curve $y = f(x)$ but has no neighboring points then it is called isolated point or, Arcnode. Ex. $y^2 = x^2(x-1)$ has isolated point $(0, 0)$.
☐ Condition: $f_{xy}^2 - f_{xx}f_{yy} < 0$ at (a, b).

[Do It Yourself] 1.68. *Find the singular points of the curve i)* $(x^2+y^2)^2 = 4(x^2 - y^2)$, *ii)* $(x^2 + y^2)^2 = 2(x^3 - 3xy^2)$, *iii)*$(x^2 + y^2)^3 = 4(x^2 - y^2)^2$, *iv)* $x^2 - x^3 + y^2 = 0$, *v)* $y(y-6) = x^2(x-2)^3 - 9$.